# More On Grover's Algorithm

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#### Abstract

The goals of this paper are to show the following. First, Grover's algorithm can be viewed as a digital approximation to the analog quantum algorithm proposed in "An Analog Analogue of a Digital Quantum Computation", by E. Farhi and S. Gutmann, Phys. Rev. A 57, 2403-2406 (1998), quant-ph/9612026. We will call the above analog algorithm the Grover-Farhi-Gutmann or GFG algorithm. Second, the propagator of the GFG algorithm can be written as a sum-over-paths formula and given a sum-over-path interpretation, i.e., a Feynman path sum/integral. We will use nonstandard analysis to do this. Third, in the semi-classical limit  $\hbar \to 0$ , both the Grover and the GFG algorithms (viewed in the setting of the approximation in this paper) must run instantaneously. Finally, we will end the paper with an open question. In "Semiclassical Shor's Algorithm", by P. Giorda, et al, Phys. Rev. A 70, 032303 (2004), quantph/0303037, the authors proposed building semi-classical quantum computers to run Shor's algorithm because the success probability of Shor's algorithm does not change much in the semi-classical limit. We ask the open questions: In the semi-classical limit, does Shor's algorithm have to run instantaneously?

### 1 Introduction

This paper attempts to answer the question: what is a quantum algorithm? In particular, what is Grover's algorithm? The views which we put forth in this paper are that quantum mechanics is an analog, continuous-time theory, and Grover's algorithm is a digitization of analog quantum mechanics. In particular, it is a digitization of the analog quantum search algorithm proposed by [7]. The digitization is performed in such a way so that the success probability of the analog search algorithm is well preserved. We are not certain that this perspective of Grover's algorithm is the correct one, but it is certainly true that the theory of quantum mechanics describes quantum evolution in continuous time.

The main goals of this paper is to show some relationships between the Grover algorithm and the analog quantum algorithm and elaborate more on the analog algorithm. We will call the analog algorithm proposed by [7] the GFG

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(Grover-Farhi-Gutmann) algorithm. The three things which we will show in this paper are the following. We will show that Grover's algorithm can be viewed as a digital approximation to the GFG algorithm. This interpretation comes about via the Lie-Trotter product formula. Second, We will use the Lie-Trotter product formula and derive a sum-over-paths formula for the propagator of the GFG algorithm. We will then give the sum-over-paths formula a Feynman path summation (integral) interpretation. This requires some nonstandard analysis. Finally, We will investigate the semi-classical limit behaviors of the GFG and the Grover algorithms (in the digital approximation setting of this paper). This will lead us to ask an open question for Shor's algorithm (for Shor's algorithm, see [19] and references within). We will assume that the reader is familiar with both the Grover and the GFG algorithms, Feynman path integrals, and a small amount of nonstandard analysis. For the readers who do not fall into the above category, we will lightly sketch the details when we go into those subjects.

In a few sentences, the GFG algorithm runs as follows. Given a Hamiltonian  $H_w = E|w\rangle\langle w|$ , drive the Hamiltonian  $H_w$  into  $H = H_w + H_s$  where <sup>1.1</sup>  $H_s = E(|s\rangle\langle s|-I)$ . Evolve the initial state  $|s\rangle$  via the evolution  $e^{-itH/\hbar}$  for some time t=t', then  $|\langle w|e^{-it'H/\hbar}|s\rangle|^2 \approx 1$ , i.e., the algorithm finds  $|w\rangle$ .

It turns out that it is possible to derive an interesting relationship between the GFG algorithm just described and Grover's algorithm. This relationship comes from the Lie-Trotter product formula and it naturally leads to the interpretation that Grover's algorithm is a digital approximation of the GFG algorithm. Using the Lie-Trotter product formula, we can break the above evolution into

$$e^{-itH_s/\hbar} = \lim_{n \to \infty} \left( e^{-itH_s/n\hbar} e^{-itH_w/n\hbar} \right)^n$$

$$\approx \left( e^{-itH_s/k\hbar} e^{-itH_w/k\hbar} \right)^k, k >> 1.$$
(1.1)

By setting t to the appropriate value, the second line of expression 1.1 becomes a k-product of the "Grover search engine"  $^{1.2}$ . In general, setting t to the appropriate value destroys the Lie-Trotter approximation in 1.1 in the sense that the error is no longer bounded by inverse powers of k. The situation is remedied by setting k to the proper value. Doing so, the success probability of finding  $|w\rangle$  in the GFG algorithm is well preserved (with error bounded by inverse powers of k) in the approximation. The analysis of the former error was previously done in [21]. We can attempt to derive a satisfying  $\mathcal{O}(1)$  bound on the former error, see [21] (we will briefly outline this in section 4). Doing this leads to the results in [21], but this is not our goal. Our goal in this paper is to concentrate on the analysis of the latter error. We wish to show that as the size of the database to be search approaches infinity, the success probability of Grover's algorithm approaches the success probability of the GFG algorithm, i.e., the error between the two probabilities is bounded by inverse powers of the size of

<sup>&</sup>lt;sup>1.1</sup>In [7],  $H_s = E|s\rangle\langle s|$ . Hence, our evolution differs from the one in [7] by a phase.

<sup>&</sup>lt;sup>1.2</sup>This term was coined by [4], see equation 2.4.

the database. Because of this, we take on the view that Grover's algorithm is a digital approximation of the GFG algorithm. In this sense, we can view Grover's algorithm as a special case of the Lie-Trotter product cut-off approximation for the GFG algorithm. By special case, we mean that the approximation is good for the success probabilities with input vector  $|s\rangle$  and output vector  $|w\rangle$ , i.e., the approximation is not global and it does not preserve the wave functions. We will show this section 4. Given the current state of the art, whether the Grover algorithm is an approximation of the GFG algorithm or the GFG algorithm is an approximation of the Grover algorithm is a matter of taste. We take on the former interpretation because quantum mechanics is an analog theory and the second line of 1.1 becomes Grover's algorithm even though it only approximates the success probability of the GFG evolution.

Those who are familiar with Feynman path integral techniques should recognize that this is an opportunity for a sum-over-paths interpretation. For more on Feynman path integrals, see [1, 2, 3, 8, 9, 13, 22]. The traditional non-relativistic quantum mechanics interpretation of the propagator  $K(\vec{x}, \vec{y}, t)$  is that it is the probability amplitude of a particle that starts at position  $\vec{x}$  at time zero and ends up at position  $\vec{y}$  at time t. The Feynman path integral interpretation of the propagator is that for any classical path that starts at  $\vec{x}$  at time zero and ends at  $\vec{y}$  at time t, there is an amplitude associated with that path and the total amplitude  $K(\vec{x}, \vec{y}, t)$  is the sum of the amplitudes of all paths. With physicists' rigor, the Feynman path integral can be derived by breaking up the evolution operator via the Lie-Trotter product formula, interpreting the limit as time slicing the physical evolution and then going to the continuous time limit. From a mathematician's point of view, this process must be done in imaginary time since in real time, there is no measure for which the Feynman path integral converges (see [2, 3]).

Extending this into quantum computing, the propagator K(j,k,t) would be the probability amplitude of starting at state j at time zero and ending up at state k at time t. A traditional Feynman path integral interpretation for this propagator is somewhat tricky because the set of "positions" for j and k is discrete and finite. A more serious problem for the propagator of the GFG algorithm is that when time sliced, the summand contains the term  $\delta_{l_m,l_{m+1}}$ , which is 1 when the state of the system is  $|l_m\rangle$  at the  $m^{th}$  time slice and is  $|l_{m+1}\rangle$  at the  $m+1^{st}$  time slice, and zero otherwise. This term is meaningless in the continuous time limit. It is for this reason that we use nonstandard analysis to give the GFG propagator a sum-over-paths interpretation. Using nonstandard analysis on Feynman path integrals is not a new concept, see [1, 14, 15, 16, 17, 18]. For more on nonstandard analysis, see [1, 5, 12, 13, 23]. The amount of nonstandard analysis that we will use is minimal. We will use the nonstandard reformulation of limits. Given a sequence  $r_n$  such that

$$\lim_{n \to \infty} r_n = r,\tag{1.2}$$

the nonstandard equivalent of equation 1.2 is the following. Given any infinite integer  $\omega \in {}^*\mathbb{N}$ , where  ${}^*\mathbb{N}$  is the nonstandard extension of the natural numbers

 $\mathbb{N}$ ,  $r_{\omega}$  is infinitesimally close to r, and  $st(r_{\omega}) = r$ . The operation st known as the standard part.

Using nonstandard analysis, we will formulate a sum-over-paths interpretation for the propagator of the GHG algorithm as follows. Fix an infinite  $\omega \in {}^*\mathbb{N}$ , time slice [0,t] into the set of discrete<sup>1.3</sup> times

$$T = \{0, t/\omega, 2t/\omega, \dots, t\}. \tag{1.3}$$

For each path<sup>1.4</sup>  $P: T \to \{1, 2, ..., N\}$ , where N is the size of the data base to be searched in Grover's algorithm, we associate an amplitude to that path. The propagator is then infinitesimally close to the sum<sup>1.5</sup> of the amplitudes of all such possible paths. Further, this has to be true for all  $\omega \in {}^*\mathbb{N}$ . We will show this in section 5.

Finally, we will investigate the semi-classical behaviors of the Grover algorithm and the GFG algorithm in the setting of this paper. We will show that in the semi-classical limit, both algorithms must run instantaneously. We must point out that if one day we figure out how to build a quantum computer to run Grover's algorithm, we do not know whether Grover's algorithm will behave this way in the semi-classical limit. This is because in the setting of this paper, we think of the Grover algorithm as a digital approximation of the GFG algorithm, which may or may not be the appropriate point of view. Recently, the authors in [10] proposed building semi-classical quantum computers for Shor's algorithm. They showed that in the semi-classical limit, the success probability of Shor's algorithm does not change much. At this point, we ask an open question. In the semi-classical limit, does Shor's algorithm have to run instantaneously?

#### $\mathbf{2}$ The Grover Algorithm

In this section, we will outline Grover's algorithm. For more details on the algorithm, see [4, 7, 11, 19]. The material in this section is taken from [4]. We are given a function  $f:\{0,\ldots,N-1\}\to\{0,1\}$  such that there exist a w with f(w) = 1 and f(a) = 0 for  $a \neq w$ . The goal is to use a quantum computer to find w.

Grover's algorithm is as follows. Let  $U_f$  be the unitary operator defined by

$$U_f|a\rangle = (-1)^{f(a)}|a\rangle = I - 2P_w, \ P_w = |w\rangle\langle w|, \tag{2.1}$$

and let

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{a=0}^{N-1} |a\rangle. \tag{2.2}$$

Define  $U_s$  by <sup>2.1</sup>.

$$U_s = 2|s\rangle\langle s| - I = 2P_s - I, \ P_s = |s\rangle\langle s|, \tag{2.3}$$
1.3 Hyper-discrete to be more precise

<sup>&</sup>lt;sup>1.4</sup>Internal path to be precise

<sup>&</sup>lt;sup>1.5</sup>Internal sum to be precise

<sup>&</sup>lt;sup>2.1</sup>In [4],  $U_s$  is defined as  $U_s = I - 2|s\rangle\langle s|$ .

and let

$$U_G = U_s U_f \tag{2.4}$$

be the "Grover search engine" <sup>2.2</sup>. Let

$$|r\rangle = \frac{1}{\sqrt{1 - (1/N)}} \left( |s\rangle - \frac{|w\rangle}{\sqrt{N}} \right),$$
 (2.5)

and let  $\mathcal{V}$  be the subspace spanned by  $\{|w\rangle, |r\rangle\}$ .

**Theorem 2.1.** The subspace V is an invariant two-dimensional subspace of  $U_G$ . Further, with respect to the orthonormal basis  $\{|w\rangle, |r\rangle\}$  in the invariant subspace V,  $U_G$  admits the unitary matrix representation

$$U = \begin{bmatrix} \frac{N-2}{N} & -\frac{2\sqrt{N-1}}{N} \\ \frac{2\sqrt{N-1}}{N} & \frac{N-2}{N} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \theta = \sin^{-1}\left(\frac{2\sqrt{N-1}}{N}\right). \quad (2.6)$$

Proof. See 
$$[4]$$
.

**Theorem 2.2.** Let  $U_G, |s\rangle, \ldots$  be as previously defined, then

$$U^{k}|s\rangle = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{k} \begin{pmatrix} \frac{1}{\sqrt{N}} \\ \sqrt{\frac{N-1}{N}} \end{pmatrix} = \begin{pmatrix} \cos(k\theta + \alpha) \\ \sin(k\theta + \alpha) \end{pmatrix}, \quad \alpha = \cos^{-1}\left(\frac{1}{\sqrt{N}}\right).$$
(2.7)

The probability of reaching the state  $|w\rangle$  after k iterations is

$$\mathfrak{P}_k = |\langle w|U^k|s\rangle|^2 = \cos^2(k\theta + \alpha). \tag{2.8}$$

Further, at  $k = \frac{\pi\sqrt{N}}{4}$ ,  $\mathcal{P}_k \approx 1$ .

Proof. See 
$$[4]$$
.

Basically, theorems 2.1 and 2.2 say that the "Grover search engine" rotates the vector  $|s\rangle$  in the subspace  $\mathcal V$  to the vector  $|w\rangle$  after k iterations.

## 3 The GFG Algorithm

In this section, we will outline the GFG algorithm. For more details on the materials in this section, see [4, 7]. The GFG algorithm is similar to the Grover algorithm in the sense that they are both search algorithms. As stated in [7], the GHG algorithm solves the following problem. Given a Hamiltonian  $H_w = E|w\rangle\langle w|$  where  $|w\rangle$  is a basis vector (the Hilbert space has dimension N), find  $|w\rangle$ .

The GFG algorithm solves the above problem as follows. Drive the Hamiltonian  $H_w$  into the Hamiltonian  $^{3.1}$ ,

$$\bar{H} = H_w + \bar{H}_s, \tag{3.1}$$

<sup>&</sup>lt;sup>2.2</sup>The term was coined by [4]

<sup>&</sup>lt;sup>3.1</sup>For our purpose, we will want  $H = H_w + H_s \equiv H_w + \bar{H}_s - EI$ . Hence the notation  $\bar{H}$  and  $\bar{H}_s$ .

where

$$\bar{H}_s = E|s\rangle\langle s|, \quad |s\rangle = \frac{1}{\sqrt{N}} \sum_{a=0}^{N-1} |a\rangle.$$
 (3.2)

Let

$$|r\rangle = \frac{1}{\sqrt{1 - (1/N)}} \left( |s\rangle - \frac{|w\rangle}{\sqrt{N}} \right),$$
 (3.3)

then, as in the Grover algorithm, the two dimensional subspace  $\mathcal{V}$  spanned by the orthonormal basis  $\{|r\rangle, |w\rangle\}$  is an invariant subspace of the evolution under  $\bar{H}$ . In other words<sup>3.2</sup>,

$$e^{-it\bar{H}/\hbar}\mathcal{V} = \mathcal{V}. \tag{3.4}$$

In the subspace V with respect to this basis, the evolution admits the matrix representation

$$e^{-it\bar{H}/\hbar} = e^{-\frac{iEt}{\hbar}} \begin{bmatrix} \cos\frac{Eyt}{\hbar} - iy\sin\frac{Eyt}{\hbar} & -\sqrt{1-y^2}i\sin\frac{Eyt}{\hbar} \\ -\sqrt{1-y^2}i\sin\frac{Eyt}{\hbar} & \cos\frac{Eyt}{\hbar} + iy\sin\frac{Eyt}{\hbar} \end{bmatrix}, y = \sqrt{\frac{1}{N}}. \quad (3.5)$$

For t > 0 and with the initial state  $|s\rangle$ , the system evolves as

$$\phi(t) = e^{-it\bar{H}/\hbar} |s\rangle$$

$$= e^{-\frac{iEt}{\hbar}} \left\{ \left[ y \cos \frac{Eyt}{\hbar} - i \sin \frac{Eyt}{\hbar} \right] |w\rangle + \sqrt{1 - y^2} \cos \frac{Eyt}{\hbar} |r\rangle \right\}.$$
(3.6)

The probability of measuring  $|w\rangle$  is then

$$\mathcal{P}(t) = |\langle w | \phi(t) |^2 = |\langle w | e^{-it\bar{H}/\hbar} | s \rangle|^2 = \sin^2 \frac{Eyt}{\hbar} + y^2 \cos^2 \frac{Eyt}{\hbar}, \ y = \sqrt{\frac{1}{N}}. \ (3.7)$$

Hence, running the system for time  $t=\frac{\pi\hbar\sqrt{N}}{2E}$  will measure  $|w\rangle$  with probability

For our purposes, we want to work with the Hamiltonian

$$H = H_w + \bar{H}_s - EI \equiv H_w + H_s = E(|w\rangle\langle w| + |s\rangle\langle s| - I) = E(P_w + P_s - I)$$
(3.8)

The evolution of the Hamiltonian H differs from the evolution of  $\bar{H}$  by a factor of  $e^{iEt/\hbar}$ .

#### Theorem 3.1. Let

$$H = H_w + \bar{H}_s - EI \equiv H_w + H_s = E(|w\rangle\langle w| + |s\rangle\langle s| - I) = E(P_w + P_s - I).$$
(3.9)

The space V is an invariant subspace of the evolution  $e^{-itH/\hbar}$ . In the orthonormal basis  $\{|w\rangle, |r\rangle\}$ ,  $e^{-itH/\hbar}$  admits the unitary matrix representation

$$e^{-itH/\hbar} = \begin{bmatrix} \cos\frac{Eyt}{\hbar} - iy\sin\frac{Eyt}{\hbar} & -\sqrt{1-y^2}i\sin\frac{Eyt}{\hbar} \\ -\sqrt{1-y^2}i\sin\frac{Eyt}{\hbar} & \cos\frac{Eyt}{\hbar} + iy\sin\frac{Eyt}{\hbar} \end{bmatrix}, y = \sqrt{\frac{1}{N}}.$$
 (3.10)
$$\frac{3.2}{\text{In the quantum computing literature, the value of } \hbar \text{ is usually taken to be 1. Because we}$$

will be taking semi-classical limits, we will not take  $\hbar = 1$ . See section 6.

For t > 0 and with the initial state  $|s\rangle$ , the system evolves as

$$\phi(t) = e^{-itH/\hbar}|s\rangle$$

$$= \left\{ \left[ y \cos \frac{Eyt}{\hbar} - i \sin \frac{Eyt}{\hbar} \right] |w\rangle + \sqrt{1 - y^2} \cos \frac{Eyt}{\hbar} |r\rangle \right\}.$$
(3.11)

The probability of measuring  $|w\rangle$  is

$$\mathcal{P}(t) = |\langle w | \phi(t) |^2 = |\langle w | e^{-itH/\hbar} | s \rangle|^2 = \sin^2 \frac{Eyt}{\hbar} + y^2 \cos^2 \frac{Eyt}{\hbar}, \ y = \sqrt{\frac{1}{N}}.$$
(3.12)

Hence, running the system for time  $t = \frac{\pi \hbar \sqrt{N}}{2E}$  will measure  $|w\rangle$  with probability 1.

*Proof.* The evolution of the Hamiltonian H is  $e^{-itH/\hbar} = e^{iEt/\hbar}e^{-it\bar{H}/\hbar}$ .

## 4 Digitizing The GFG Algorithm

In this section, we will digitize the GFG algorithm. The philosophy which we adapt here is that we are aware of the GFG algorithm and have the Lie-Trotter product formula at our disposal. We will see that under this philosophy, digitizing the GFG algorithm naturally leads to the Grover algorithm. For those who are familiar with Feynman path integral methods, it is also possible to view the Grover algorithm as a by product of giving the GFG algorithm a Feynman path integral interpretation. Of course this point of view is hindsight since the development of the two algorithms came in the reverse order and we are aware of Grover's algorithm. It is not surprising that this is hindsight since Feynman path integral methods are in general more complex than operator methods <sup>4.1</sup>. We take on the approach in this section because historically, quantum mechanics is a time dependent analog theory and because it is pedagogical in answering the question: what is a quantum algorithm?

We start with the Lie-Trotter produce formula. For our purposes, we only need the finite dimensional version, which is the Lie product formula, see [20].

**Theorem 4.1.** Let A and B be self-adjoint, finite-dimensional matrices, then

$$e^{-i\eta(A+B)} = \lim_{n \to \infty} \left[ e^{-i\eta A/n} e^{-i\eta B/n} \right]^n. \tag{4.1}$$

Furthermore, for any k,

$$||e^{-i\eta(A+B)} - [e^{-i\eta A/k}e^{-i\eta B/k}]^k|| \le \frac{\eta^2 \mathcal{O}(||[A,B]||)}{k},$$
 (4.2)

where [A, B] = AB - BA.

<sup>&</sup>lt;sup>4.1</sup>For example, it was many years after Feynman's formulation of the Feynman path integral before the hydrogen atom path integral was solved (with physicist's rigor). Feynman himself was not able to solve the hydrogen atom path integral. See [13] and references within.

*Proof.* Let  $S_k = e^{-i\eta(A+B)/k}$ , and  $T_k = e^{-i\eta A/k}e^{-i\eta B/k}$ . Then

$$||S_k^k - T_k^k|| = \left| \left| \sum_{m=0}^{k-1} S_k^m (S_k - T_k) T_k^{k-1-m} \right| \right|$$

$$\leq k ||S_k^m|| ||T_k^{k-1-m}|| ||(S_k - T_k)|| = k ||(S_k - T_k)||.$$

$$(4.3)$$

Further, using the Campbell-Baker-Hausdorff approximation yields

$$||(S_k - T_k)|| = \frac{\eta^2 \mathcal{O}(||[A, B]||)}{k^2}.$$
 (4.4)

Expressions 4.3 and 4.4 imply

$$||e^{-i\eta(A+B)} - [e^{-i\eta A/k}e^{-i\eta B/k}]^k|| = ||S_k^k - T_k^k|| \le \frac{\eta^2 \mathcal{O}(||[A, B]||)}{k} \to 0. \quad \Box$$
(4.5)

Theorem 4.1 is the first step towards a Feynman path integral interpretation for the GFG algorithm. We will derive a sum-over-paths interpretation in section 5. We continue with our attempt to digitize the GHG algorithm.

**Proposition 4.1.** Let P be a projection operator, then

$$e^{-i\pi P} = I - 2P \tag{4.6}$$

*Proof.* Power expanding and using the fact that P is a projection, we get

$$e^{-i\pi P} = \sum_{n=0}^{\infty} \frac{(-i\pi P)^n}{n!} = I + \sum_{n=1}^{\infty} \frac{(-i\pi)^n}{n!} P = I + (e^{-i\pi} - 1)P = I - 2P. \quad \Box \quad (4.7)$$

**Proposition 4.2.** Let  $P_w = |w\rangle\langle w|$ , then  $U_f = e^{-i\pi P_w}$ .

*Proof.* This is an application of proposition 4.1 by writing  $U_f = I - 2P_w$ .

**Proposition 4.3.** Let  $P_s = |s\rangle\langle s|$ , then  $U_s = e^{-i\pi(P_s - I)}$ .

*Proof.* This is an application of proposition 4.1 by writing 
$$U_s = -(I - 2P_s) = e^{i\pi I}e^{-i\pi P_s} = e^{-i\pi(P_s - I)}$$
.

We are now ready to digitize the GHG algorithm presented in theorem 3.1. The Hamiltonian which we use for the GFG algorithm is  $H = E(P_s - I + P_w)$ , which produces the evolution  $e^{-itE(P_s - I + P_w)/\hbar}$ . Using the Lie-Trotter product formula on this evolution yields

$$e^{-itE(P_s - I + P_w)/\hbar} = \lim_{n \to \infty} \left( e^{-itE(P_s - I)/n\hbar} e^{-itEP_w/n\hbar} \right)^n. \tag{4.8}$$

Further, for large k, we can approximate the evolution via

$$e^{-itE(P_s-I+P_w)/\hbar} \approx \left(e^{-itE(P_s-I)/k\hbar}e^{-itEP_w/k\hbar}\right)^k,$$
 (4.9)

where the error in the approximation is of order  $\frac{t^2E^2}{\hbar^2k}$ . For  $k >> \frac{t^2E^2}{\hbar^2}$ , this approximation works well. Suppose we let  $\frac{tE}{\hbar} = k\pi$ , then according to propositions 4.2 and 4.3, he right hand side of approximation 4.9 becomes  $(U_sU_f)^k$ , which is a k-product of the "Grover search engine". With  $\frac{tE}{\hbar} = k\pi$ , the error bound in approximation 4.9 is no longer in inverse powers of k, it is O(1). This can be seen as follows. The Lie-Trotter product formula tells us that the error is bounded by

$$\frac{t^2 \mathcal{O}(||[P_s - I, P_w]||)}{k}. (4.10)$$

Further,

$$||[P_s - I, P_w]|| = \mathcal{O}\left(\frac{1}{\sqrt{N}}\right),\tag{4.11}$$

and for the GFG and the Grover algorithms,

$$t = \mathcal{O}\left(\sqrt{N}\right), k = \mathcal{O}\left(\sqrt{N}\right).$$
 (4.12)

Hence, 4.10 becomes  $\mathcal{O}(1)$ . This result, which is not our goal, was obtained by [21]. Our goal here is to obtain error bounds in inverse powers of k. Thus, in the asymptotic limit on the size of the database to be searched, the success probability of the Grover algorithm approaches the success probability of the GFG algorithm.

As far as running the GFG algorithm is concerned, we are interested in the probability

$$\mathcal{P}(t) = |\langle w|e^{-itH/\hbar}|s\rangle|^2 = \sin^2\frac{Eyt}{\hbar} + y^2\cos^2\frac{Eyt}{\hbar}, \ y = \sqrt{\frac{1}{N}}.$$
 (4.13)

Thus, we settle for values of t's and k's such that the following three properties are satisfied. First,  $t = \mathcal{O}\left(\sqrt{N}\right)$ . Second, the right hand side of 4.9 becomes a k-product of the "Grover search engine". Third, with those values of t's and k's, the probabilities given by equations 4.13 and 2.8 are close to each other with error bounded by inverse powers of k. In other words, rather than globally digitize the GFG evolution operator, we settle for digitizing the operator in such a way so that the success probability of the GFG algorithm is well preserved. More simply, we wish to find correct combinations of t's and t's so that

$$1 \approx \mathcal{P}(t) = |\langle w|e^{-itH/\hbar}|s\rangle|^2 = \left|\langle w|\lim_{n\to\infty} \left(e^{-itE(P_s-I)/n\hbar}e^{-itEP_w/k\hbar}\right)^n|s\rangle\right|^2$$

$$\approx \left|\langle w|\left(e^{-itE(P_s-I)/k\hbar}e^{-itEP_w/k\hbar}\right)^k|s\rangle\right|^2 = \left|\langle w|\left(U_sU_f\right)^k|s\rangle\right|^2,$$
(4.14)

where the error of the second approximation is bounded by inverse powers of k.

**Theorem 4.2.** Let  $l, m \in \mathbb{N}$  be such that  $l, m << \sqrt{N}$ , m is odd, and

$$\frac{4\pi l}{\pi - 2} = m + \epsilon,\tag{4.15}$$

where  $|\epsilon| \approx 0$ . Let  $k = \left[\frac{2\pi l \sqrt{N}}{\pi - 2}\right]$  be the nearest integer to  $\frac{2\pi l \sqrt{N}}{\pi - 2}$ , and let  $t = \frac{k\pi \hbar}{E}$ . Finally, let

$$\delta = \max\left\{\epsilon, \sqrt{\frac{1}{N}}\right\}. \tag{4.16}$$

Then,

$$\left(e^{-itE(P_s-I)/k\hbar}e^{-itEP_w/k\hbar}\right)^k = \left(U_sU_f\right)^k,\tag{4.17}$$

$$\mathcal{P}(t) = |\langle w|e^{-itH/\hbar}|s\rangle|^2$$

$$= \sin^2\left(\frac{Et}{\sqrt{N}\hbar}\right) + \frac{1}{N}\cos^2\left(\frac{Et}{\sqrt{N}\hbar}\right) = 1 + \mathcal{O}\left(\delta^2\right),$$
(4.18)

and

$$\mathfrak{P}_{k} = \left| \left\langle w \right| \left( e^{-itE(P_{s} - I)/k\hbar} e^{-itEP_{w}/k\hbar} \right)^{k} \left| s \right\rangle \right|^{2} \\
= \left| \left\langle w \right| \left( U_{s}U_{f} \right)^{k} \left| s \right\rangle \right|^{2} = \cos^{2}\left( k\theta + \alpha \right) = \mathfrak{P}(t) + \mathfrak{O}\left( \frac{1}{N} \right) \tag{4.19}$$

*Proof.* Equation 4.17 follows from propositions 4.2 and 4.3. Recall that (see [4])

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{N}}\right) = \frac{\pi}{2} + \mathcal{O}\left(\sqrt{\frac{1}{N}}\right),\tag{4.20}$$

$$\theta = \sin^{-1}\left(\frac{2\sqrt{N-1}}{N}\right) = 2\sqrt{\frac{1}{N}} + O\left(\frac{1}{N^{3/2}}\right).$$
 (4.21)

With the above values of k, we have

$$k\theta = \frac{2\pi l\sqrt{N}}{\pi - 2} 2\sqrt{\frac{1}{N}} + \mathcal{O}\left(\sqrt{\frac{1}{N}}\right) = \frac{4\pi l}{\pi - 2} + \mathcal{O}\left(\sqrt{\frac{1}{N}}\right). \tag{4.22}$$

Further,

$$\frac{Et}{\sqrt{N}\hbar} = \frac{k\pi}{\sqrt{N}} = \frac{2\pi l\sqrt{N}}{\pi - 2} \frac{\pi}{\sqrt{N}} + \mathcal{O}\left(\sqrt{\frac{1}{N}}\right) = \frac{2\pi^2 l}{\pi - 2} + \mathcal{O}\left(\sqrt{\frac{1}{N}}\right). \tag{4.23}$$

Then,

$$k\theta - \frac{Et}{\sqrt{N}\hbar} = \frac{4\pi l}{\pi - 2} - \frac{2\pi^2 l}{\pi - 2} + \mathcal{O}\left(\sqrt{\frac{1}{N}}\right) = -2\pi l + \mathcal{O}\left(\sqrt{\frac{1}{N}}\right). \tag{4.24}$$

Hence,

$$\mathfrak{P}_{k} = \left| \left\langle w \right| \left( e^{-itE(P_{s}-I)/k\hbar} e^{-itEP_{w}/k\hbar} \right)^{k} \left| s \right\rangle \right|^{2} \\
= \left| \left\langle w \right| \left( U_{s}U_{f} \right)^{k} \left| s \right\rangle \right|^{2} = \cos^{2}\left( k\theta + \alpha \right) \\
= \cos^{2}\left[ \frac{Et}{\sqrt{N}\hbar} - 2\pi l + \frac{\pi}{2} + \mathcal{O}\left(\sqrt{\frac{1}{N}}\right) \right] \\
= \sin^{2}\left( \frac{Et}{\sqrt{N}\hbar} \right) + \mathcal{O}\left( \frac{1}{N} \right) \\
= \mathfrak{P}(t) + \mathcal{O}\left( \frac{1}{N} \right). \tag{4.25}$$

As for P(t), we have

$$P(t) = \sin^{2}\left(\frac{Et}{\sqrt{N}\hbar}\right) + \frac{1}{N}\cos^{2}\left(\frac{Et}{\sqrt{N}\hbar}\right) = \sin^{2}\left(\frac{Et}{\sqrt{N}\hbar}\right) + \mathcal{O}\left(\frac{1}{N}\right)$$

$$= \sin^{2}\left(\frac{2\pi^{2}l}{\pi - 2} + \mathcal{O}\left(\sqrt{\frac{1}{N}}\right)\right) + \mathcal{O}\left(\frac{1}{N}\right)$$

$$= \sin^{2}\left(\frac{\pi}{2}\frac{4\pi l}{\pi - 2} + \mathcal{O}\left(\sqrt{\frac{1}{N}}\right)\right) + \mathcal{O}\left(\frac{1}{N}\right)$$

$$= \sin^{2}\left(\frac{\pi}{2}m + \mathcal{O}(\delta)\right) + \mathcal{O}\left(\frac{1}{N}\right).$$

$$(4.26)$$

since m is odd, 4.26 becomes

$$P(t) = 1 + \mathcal{O}(\delta^2). \quad \Box \tag{4.27}$$

In light of theorem 4.2, we can interpret Grover's algorithm as a digitization of the GFG algorithm via the Lie product formula. While the wave function is not preserved under the digitization, the success probability is well preserved. We can also view the "Grover search engine" as  $e^{-itE(P_s-I)/k\hbar}e^{-itEP_w/k\hbar}$  with t and k as specified in theorem 4.2. Obviously, this view might only be useful for theoretical analysis since it would not be efficient to run Grover's algorithm by running  $t = \mathcal{O}\left(\sqrt{N}\right)$  and then  $k = \mathcal{O}\left(\sqrt{N}\right)$  copies of the "Grover search engine".

We end this section with an numerical example of theorem 4.2. Suppose

$$N = 2^{30}, \text{ and } l = 1. \text{ Then,}$$

$$\frac{4\pi l}{\pi - 2} \approx 11 + .00775358 \equiv m + \epsilon,$$

$$\delta = \max\left\{\epsilon, \frac{1}{\sqrt{N}}\right\} = \max\left\{.00775358, \frac{1}{2^{15}}\right\} = \epsilon,$$

$$\alpha = \cos^{-1}\left(\frac{1}{2^{15}}\right) \approx 1.57076581,$$

$$\theta = \sin^{-1}\left(\frac{2\sqrt{2^{30} - 1}}{2^{30}}\right) \approx .00006104,$$

$$k = \left[\frac{2\pi 2^{15}}{\pi - 2}\right] = 180351,$$

$$\frac{Et}{\sqrt{N}\hbar} = \frac{k\pi}{\sqrt{N}} \approx 17.29093889.$$

$$(4.28)$$

With the values given in 4.28, we have

$$P(t) \approx \sin^2 (17.29093889) + \frac{1}{2^{30}} \cos^2 (17.29093889)$$
  
 
$$\approx .99985167 = 1 + \mathcal{O}(\epsilon^2) = 1 + \mathcal{O}(.00006012).$$
 (4.29)

Lastly,

$$P_k = \cos^2(k\theta + \alpha) \approx \cos^2(180351 * .00006104 + 1.57076581)$$
  
 
$$\approx .99983048 = .99985167 + \mathcal{O}(.00003052) = P(t) + \mathcal{O}\left(\frac{1}{N}\right). \tag{4.30}$$

Notice that in the example, we did not specify the value of E. Hence, we do not have a numerical value for t.

## 5 Sum-Over-Paths Interpretation

In this section, we will derive a sum-over-paths (Feynman path integral) interpretation for the GFG algorithm. Recently, sum-over-paths techniques were applied to quantum algorithms, see [6]. The sum-over-paths interpretation of the work in [6] is somewhat unconventional compared to traditional Feynman path integral methods (see [8, 9, 13, 22]) in the sense that the number of paths in [6] are finite. It is the goal of this section to derive a sum-over-paths interpretation for the GFG algorithm using more traditional methods. In traditional Feynman path integrals, breaking up the evolution operator  $e^{-it(H_1+H_2)}$  via the Lie-Trotter product formula is the first step towards the derivation of the Feynman path integral.

We are interested in deriving a sum-over-paths formula for the propagator of the GFG algorithm. For any quantum states  $|x\rangle$  and  $|y\rangle$ , we can write

$$\langle x|e^{-itH/\hbar}|y\rangle = \sum_{j,k} \langle x|j\rangle\langle j|e^{-itH/\hbar}|k\rangle\langle k|y\rangle. \tag{5.1}$$

The propagator K(j, k, t) is defined by

$$K(j,k,t) \equiv \langle j|e^{-itH/\hbar}|k\rangle.$$
 (5.2)

The traditional non-relativistic quantum mechanics interpretation of the propagator  $K(\vec{x}, \vec{y}, t)$  is that it is the probability amplitude of a particle that starts at position  $\vec{x}$  at time zero and ends up at position  $\vec{y}$  at time t. The Feynman path integral interpretation of the propagator is that for any classical path from  $\vec{x}$  to  $\vec{y}$  at time zero to time t, an amplitude is assigned to that path and the total amplitude  $K(\vec{x}, \vec{y}, t)$  is the sum of all such path amplitudes. We are interested in obtaining a similar interpretation for the propagator in equation 5.2.

**Proposition 5.1.** Let  $|l\rangle$  be a basis vector of the computational basis, then

$$e^{\frac{-itE|w\rangle\langle w|}{n\hbar}}|l\rangle = e^{\frac{-itEf(l)}{n\hbar}}|l\rangle. \tag{5.3}$$

*Proof.* We have

$$e^{\frac{-itE|w\rangle\langle w|}{n\hbar}}|l\rangle = |l\rangle + \sum_{r=1}^{\infty} \left(\frac{-itE}{n\hbar}\right)^r \frac{1}{r!} P_w|l\rangle = |l\rangle + \left(e^{\frac{-itE}{n\hbar}} - 1\right) \delta_{w,l}|l\rangle$$

$$= \left[1 + \left(e^{\frac{-itE}{n\hbar}} - 1\right) f(l)\right]|l\rangle. \tag{5.4}$$

Notice that

$$1 + \left(e^{\frac{-itE}{n\hbar}} - 1\right)f(l) = \begin{cases} e^{\frac{-itE}{n\hbar}} & \text{if } l = w\\ 1 & \text{if } l \neq w \end{cases}$$
 (5.5)

Hence the proposition follows.

**Proposition 5.2.** Let  $|l\rangle$ ,  $|m\rangle$  be basis vectors of the computational basis, then

$$\langle l|e^{\frac{-itE(|s\rangle\langle s|-I)}{n\hbar}}|m\rangle = e^{\frac{-itE(-1)}{n\hbar}} \left[\delta_{l,m} + \frac{1}{N} \left(e^{\frac{-itE}{n\hbar}} - 1\right)\right]. \tag{5.6}$$

*Proof.* Commuting the identity matrix and then Expanding the exponent yields

$$\langle l|e^{\frac{-itE(|s\rangle\langle s|-I)}{n\hbar}}|m\rangle = e^{\frac{-itE(-1)}{n\hbar}} \left[\delta_{l,m} + \left(e^{\frac{-itE}{n\hbar}} - 1\right)\langle l|s\rangle\langle s|m\rangle\right]$$

$$= e^{\frac{-itE(-1)}{n\hbar}} \left[\delta_{l,m} + \frac{1}{N}\left(e^{\frac{-itE}{n\hbar}} - 1\right)\right]. \quad \Box$$
(5.7)

**Proposition 5.3.** Let  $l_0 = k$  and  $l_n = j$ , then The propagator in equation 5.2is given by

$$K(j,k,t) = \lim_{n \to \infty} \sum_{l_1,l_2,\dots,l_{n-1}} \prod_{m=0}^{n-1} e^{\frac{-itE(f(l_m)-1)}{n\hbar}} \left[ \delta_{l_m,l_{m+1}} + \frac{\left(e^{\frac{-itE}{n\hbar}} - 1\right)}{N} \right] = \lim_{n \to \infty} \sum_{l_1,l_2,\dots,l_{n-1}} \exp \frac{-itE\sum_{m=0}^{n-1} \left(f(l_m) - 1\right)}{n\hbar} \prod_{m=0}^{n-1} \left[ \delta_{l_m,l_{m+1}} + \frac{\left(e^{\frac{-itE}{n\hbar}} - 1\right)}{N} \right]$$
(5.8)

*Proof.* Using the Lie-Trotter produce formula, equation 5.2 can be written as

$$K(j,k,t) = \lim_{n \to \infty} \langle j | \left[ e^{\frac{-itE(|s\rangle\langle s|-I)}{n\hbar}} e^{\frac{-itE|w\rangle\langle w|}{n\hbar}} \right]^n |k\rangle.$$
 (5.9)

For each of the n products, inserting the identity

$$I = \sum_{l_{\alpha}} |l_{\alpha}\rangle\langle l_{\alpha}|, \quad \alpha = 1, 2, \dots n - 1$$
(5.10)

and denoting

$$k \equiv l_0, \quad j \equiv l_n \tag{5.11}$$

yields

$$K(j,k,t) = \lim_{n \to \infty} \sum_{l_1,l_2,\dots,l_{n-1}} \prod_{m=0}^{n-1} \langle l_{m+1} | e^{\frac{-itE(|s\rangle\langle s|-I)}{n\hbar}} e^{\frac{-itE|w\rangle\langle w|}{n\hbar}} | l_m \rangle.$$
 (5.12)

The result follows by applying propositions 5.1 and 5.2 to the last equation.  $\square$ 

Unfortunately, in the continuous limit, a sum-over-paths interpretation of proposition 5.3 is problematic. The expression  $\delta_{l_m,l_{m+1}}$  has the value 1 if at time mt/n, the state of the system is  $|l_m\rangle$  and at time (m+1)t/n, the state of the system is  $|l_{m+1}\rangle$ . In the limit to continuous time,  $\delta_{l_m,l_{m+1}}$  becomes meaningless.

We proposed a nonstandard analysis formulation of proposition 5.3 which allows us to give proposition 5.3 a sum-over-paths interpretation. For details on nonstandard analysis, see [1, 5, 12, 23] and references within.

**Theorem 5.1.** Let  $\omega \in {}^*\mathbb{N}$  be an infinite nonstandard natural number, then the propagator for the GFG algorithm is given by

$$K(j, k, t) = st \left\{ \sum_{l_1, l_2, \dots, l_{\omega - 1}} \exp \left[ \frac{-itE \sum_{m=0}^{\omega - 1} (f(l_m) - 1)}{\omega \hbar} \right] \prod_{m=0}^{\omega - 1} \left[ \delta_{l_m, l_{m+1}} + \frac{1}{N} \left( e^{\frac{-itE}{\omega \hbar}} - 1 \right) \right] \right\}.$$
(5.13)

*Proof.* This is an application of nonstandard analysis's formulation of limits to proposition 5.3.

The interpretation of theorem 5.1 is as follows. First, fix an infinite  $\omega \in {}^*\mathbb{N}$  and time slice [0,t] into  $\omega+1$  number of steps  $T=\{0,t/\omega,2t/\omega,\ldots,t\}$ . Notice that  $\epsilon\equiv t/\omega$  is infinitesimal. For any path

$$P: T \to \{1, 2, \dots, N\},$$
 (5.14)

we associate the amplitude

$$\exp\left[\frac{-itE\sum_{m=0}^{\omega-1}\left(f(P(m\epsilon))-1\right)}{\omega\hbar}\right]\prod_{m=0}^{\omega-1}\left[\delta_{P(m\epsilon),P((m+1)\epsilon)}+\frac{1}{N}\left(e^{\frac{-itE}{\omega\hbar}}-1\right)\right]$$
(5.15)

to P. The propagator for GFG algorithm evolution is then infinitesimally close to the sum of the amplitudes of all possible paths P. Notice that for all fixed infinite  $\omega \in {}^*\mathbb{N}$ , the amplitude of any path has to be of the form given in equation 5.15.

### 6 Semi-classical Limits

In this last section, we will investigate the semi-classical limit behavior of the GFG and the Grover algorithms within the context of this paper and discuss some open problems.

One way to obtain the semi-classical limit of a quantum system is by taking the limit  $\hbar \to 0$ . In the literature of quantum computing,  $\hbar$  is usually set to the value of 1. Throughout this paper, we have included  $\hbar$  in all the equations. According to section 4, the GFG algorithm is obtained by setting  $t = \frac{k\pi\hbar}{E}$ . Further, the two unitary operators  $U_f$  and  $U_s$  were also obtained from the Lie-Trotter product approximation by setting  $t = \frac{k\pi\hbar}{E}$ . Hence, in the context of this paper, in the semi-classical limit, both the GFG and the Grover algorithm must run instantaneously. This is because  $t \to 0$  as  $\hbar \to 0$ . It is not clear if this has any meaning in terms of the standard interpretation of Grover's algorithm in the literature. In the literature, the unitary operators  $U_s$  and  $U_f$  are time independent which could unrealistic since quantum mechanics is evolved by time dependent evolution operators. If we view decoherence due to interactions with the environment as the quantum system becoming classical, then we might consider modeling decoherence by  $\hbar \to 0$ . If we model decoherence this way, then the semi-classical limit behavior of the GFG and the Grover algorithm tells us that the algorithms must run instantaneously before the systems become decoherent.

One could argue that semi-classical limit behaviors are not relevant to quantum computing, but recently there has been work done on the semi-classical limit behaviors of Shor's algorithm, see [10]. The authors in [10] proposed that it might be worthwhile to consider building quantum computers to run Shor's algorithm by using semi-classical devices. The authors showed that in the semi-classical limit, Shor's algorithm's success probability is not too severely modified. It would be interesting to see whether techniques similar to the ones used in this paper can be applied to Shor's algorithm and if in the semi-classical limit, Shor's algorithm must run instantaneously.

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